6 | Some Applications

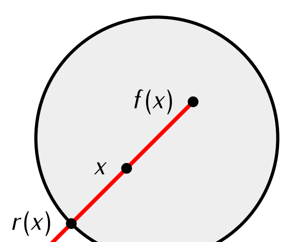
In this chapter we will use the computations of the fundamental group we completed so far to obtain a few interesting results. We start with the fact that was already mentioned in Chapter 1.

6.1 Proposition. *The circle S^1 is not a retract of the disc D^2.*

Proof. See the proof of Proposition 1.2. ∎

6.2 Brouwer Fixed Point Theorem. *For each map $f\colon D^2 \to D^2$ there exists a point $x_0 \in D^2$ such that $f(x_0) = x_0$.*

Proof. We argue by contradiction. Assume that $f\colon D^2 \to D^2$ is a continuous function such that $f(x) \neq x$ for all $x \in D^2$. Define a function $r\colon D^2 \to S^1$ as follows. For a point $x \in D^2$ let $L_x \subseteq \mathbb{R}^2$ be the half-line that begins at $f(x)$ and that passes through the point x. This half-line intersects with S^1 at exactly one point. We set $r(x)$ to be the point of intersection:

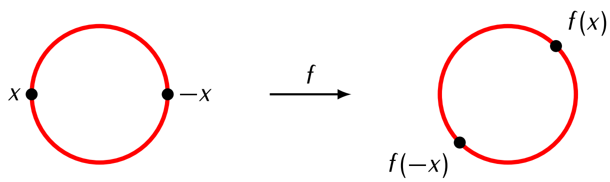


One can check that r is a continuous function (exercise). Since for $x \in S^1$ we have $r(x) = x$ the function r is a retraction of D^2 onto S^1. This contradicts Proposition 6.1.

∎

6.3 Borsuk-Ulam Theorem. *For each map $f\colon S^2 \to \mathbb{R}^2$ there exists $x \in S^2$ such that $f(x) = f(-x)$.*

6.4 Lemma. *Let $f\colon S^1 \to S^1$ be a function such that $f(-x) = -f(x)$ for all $x \in S^1$:*



For any $x_0 \in S^1$ the homomorphism $f_*\colon \pi_1(S^1, x_0) \to \pi_1(S^1, f(x_0))$ is non-trivial.

Proof. Exercise. □

Proof of Theorem 6.3. We argue by contradiction. Assume that $f\colon S^2 \to \mathbb{R}^2$ is a function such that $f(x) \neq f(-x)$ for all $x \in S^2$ and let $g\colon S^2 \to S^1$ be the function given by

$$g(x) = \frac{f(x) - f(-x)}{\big\|f(x) - f(-x)\big\|}$$

Notice that by assumption $f(x) - f(-x) \neq 0$ for all $x \in S^2$, so g is well defined. Notice also that $g(-x) = -g(x)$ for all $x \in S^2$. Let $j\colon S^1 \to S^2$ be the inclusion of S^1 onto the equator of S^2: $j(x, y) = (x, y, 0)$. The composition $gj\colon S^1 \to S^1$ satisfies the assumption of Lemma 6.4, so for any $x_0 \in S^1$ the homomorphism

$$(gj)_*\colon \pi_1(S^1, x_0) \to \pi_1(S^1, gj(x_0))$$

is non-trivial. On the other hand we have $gj = g|_{S^2_+} j$ where $g|_{S^2_+}$ is the restriction of g to the upper hemisphere $S^2_+ \subseteq S^2$. This gives a commutative diagram:

$$\begin{tikzcd}
\pi_1(S^1, x_0) \arrow[rr, "(gj)_*"] \arrow[dr, "j_*"'] & & \pi_1(S^1, gj(x_0)) \\
& \pi_1(S^2_+, j(x_0)) \arrow[ur, "(g|_{S^2_+})_*"'] &
\end{tikzcd}$$

Since $S^2_+ \cong D^2$ by Proposition 5.3 we get that the group $\pi_1(S^2_+)$ is trivial, and so $(gj)_*$ is the trivial homomorphism. Thus we obtain a contradiction. □

6.5 Corollary. *There does not exist an embedding of S^2 into $\mathbb{R}^2$.*

Proof. An embedding $S^2 \to \mathbb{R}^2$ would be a 1-1 map which by Theorem 6.3 does not exist. □

6.6 Corollary. *If $A_1, A_2, A_3 \subseteq S^2$ are closed sets such that $A_1 \cup A_2 \cup A_3 = S^2$ then one of these sets contains a pair of antipodal points $\{x, -x\}$.*

Proof. For $x \in S^2$ let $d_i(x)$ denote the distance from x to the set A_i:

$$d_i(x) = \inf\{\|x - y\| \mid y \in A_i\}$$

The function $d_i\colon S^2 \to \mathbb{R}$ is continuous. Also, since A_i is closed we have $d_i(x) = 0$ if and only if $x \in A_i$. Consider the function $d_{12}\colon S^2 \to \mathbb{R}^2$ given by $d_{12}(x) = (d_1(x), d_2(x))$. By Theorem 6.3 there exists a point $x_0 \in S^2$ such that $d_{12}(x_0) = d_{12}(-x_0)$, i.e. $d_1(x_0) = d_1(-x_0)$ and $d_2(x_0) = d_2(-x_0)$. It follows that if $d_1(x_0) = 0$ then also $d_1(-x_0) = 0$, and so $\{x_0, -x_0\} \subseteq A_1$. Likewise, if $d_2(x_0) = 0$ then $\{x_0, -x_0\} \subseteq A_2$. If $d_1(x_0) > 0$ and $d_2(x_0) > 0$ then $\{x_0, -x_0\} \subseteq S^2 \smallsetminus (A_1 \cup A_2) \subseteq A_3$. □

6.7 The Fundamental Theorem of Algebra. *If $P(x)$ is a polynomial with coefficients in $\mathbb{C}$ and $\deg P(x) > 0$ then $P(z_0) = 0$ for some $z_0 \in \mathbb{C}$.*

Proof. We start with a few preliminary observations. We will consider S^1 as the subspace of the complex plane:

$$S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$$

and we will take $1 \in \mathbb{C}$ as the basepoint of S^1. Consider the degree isomorphism $\deg\colon \pi_1(S^1, 1) \to \mathbb{Z}$. Notice that if $\omega_n\colon [0,1] \to S^1$ is the loop given by $\omega_n(s) = e^{2\pi i n s}$ then $\deg([\omega_n]) = n$.

We will prove Theorem 6.7 by contradiction. Assume that $P(x) = x^n + a_{n-1}x^n + \cdots + a_0$ is a polynomial with complex coefficients such that $n > 0$ and that $P(z) \neq 0$ for all $z \in \mathbb{C}$. For $r \geq 0$ let $\sigma_r\colon [0,1] \to S^1$ be a loop based at $1 \in S^1$ given by

$$\sigma_r(s) = \frac{P(re^{2\pi i s})/P(r)}{\left\|P(re^{2\pi i s})/P(r)\right\|}$$

We have $[\sigma_r] \in \pi_1(S^1, 1) \cong \mathbb{Z}$.

Claim 1. For each $r \geq 0$ we have $[\sigma_r] = 0$.

For $r = 0$ this is true since σ_0 is the constant loop. For $r > 0$ the map $h\colon [0,1] \times [0,1] \to S^1$ defined by $h(s,t) = \sigma_{tr}(s)$ gives a homotopy between σ_0 and σ_r so $[\sigma_r] = [\sigma_0] = 0$.

Claim 2. If $r > \max\left\{1, \left\|a_{n-1}\right\| + \cdots + \left\|a_0\right\|\right\}$ then $[\sigma_r] \neq 0$.

Indeed, assume that r satisfies the assumption of Claim 2. If $\|z\| = r$ then

$$\begin{aligned}
\|z^n\| &= r \cdot \|z^{n-1}\| \\
&\geq (\|a_{n-1}\| + \cdots + \|a_0\|) \cdot \|z^{n-1}\| \\
&\geq \|a_{n-1}\| \cdot \|z^{n-1}\| + \cdots + \|a_1\| \cdot \|z\| + \|a_0\| \\
&\geq \|a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\|
\end{aligned}$$

For $t \in [0,1]$ take the polynomial $P_t(x) = x^n + t(a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)$. The inequality above shows that $P_t(z) \neq 0$ for all $z \in \mathbb{C}$ such that $\|z\| = r$. Define $h\colon [0,1] \times [0,1] \to S^1$ by

$$h(s,t) = \frac{P_t(re^{2\pi i s})/P_t(r)}{\left\|P_t(re^{2\pi i s})/P_t(r)\right\|}$$

The map h gives a path homotopy between σ_r and the loop ω_n defined above. Therefore $\deg([\sigma_r]) = \deg([\omega_n]) = n \neq 0$.

Since Claim 1 and Claim 2 contradict each other we are done. $\square$

Exercises to Chapter 6

E6.1 Exercise. Prove Lemma 6.4

E6.2 Exercise. Let $f\colon S^2 \to \mathbb{R}^2$ be a function such that $f(-x) = -f(x)$ for all $x \in S^2$. Show that there exists a point $x_0 \in S^2$ such that $f(x_0) = 0$

E6.3 Exercise. Let $\omega, \tau\colon [0,1] \to [0,1] \times [0,1]$ be paths in the square such that $\omega(0) = (0,0)$, $\omega(1) = (1,1)$, $\tau(0) = (1,0)$, and $\tau(1) = (0,1)$. Show that $\omega(s) = \tau(t)$ for some $s, t \in [0,1]$. (Hint: use Brouwer fixed point theorem.)