5 | First Computations

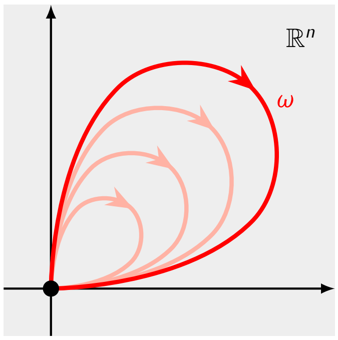
In this chapter we describe some basic examples of computations of the fundamental group. Later on we will see that these examples and a few additional tools let us calculate fundamental groups of many spaces.

5.1 Proposition. *If $X = \{*\}$ is a space consisting of only one point then $\pi_1(X)$ is the trivial group.*

Proof. It is enough to notice that the only loop in X is the constant loop. □

5.2 Proposition. *For any $n \geq 1$ the group $\pi_1(\mathbb{R}^n)$ is trivial.*

Proof. Choose $0 \in \mathbb{R}^n$ as the basepoint. Let $\omega\colon [0,1] \to \mathbb{R}^n$ be a loop based at 0. We need to show that ω is homotopic to the constant loop c_0. Such homotopy $h\colon [0,1] \times [0,1] \to \mathbb{R}^n$ is given by $h(s,t) = t \cdot \omega(s)$.



□

Let $D^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 \leq 1\}$ be the n-dimensional closed unit disc. Using the the same argument as above we obtain:

5.3 Proposition. *For any $n \geq 1$ the group $\pi_1(D^n)$ is trivial.*

5.4 Note. As we have seen before homeomorphic spaces have isomorphic fundamental groups (3.17). The above calculations show that the converse is not true, e.g. $\mathbb{R}^n \ncong D^n$ for $n \geq 1$ but $\pi_1(\mathbb{R}^n) \cong \pi_1(D^n)$.

5.5 Definition. A space X is *simply connected* if it is path connected and $\pi_1(X)$ is trivial.

For example $\{*\}$, $\mathbb{R}^n$, and D^n are simply connected spaces.

5.6 Proposition. *A space X is simply connected if and only if X is path connected and for any two paths $\omega, \tau\colon [0,1] \to X$ satisfying $\omega(0) = \tau(0)$ and $\omega(1) = \tau(1)$ we have $\omega \simeq \tau$.*

Proof. Exercise. □

Our next goal will be to show that the fundamental group is not always trivial:

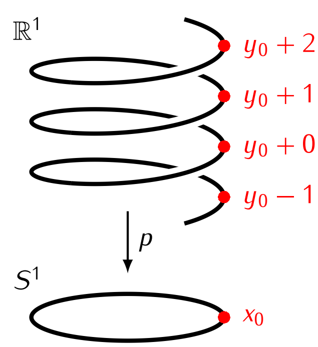
5.7 Theorem. $\pi_1(S^1) \cong \mathbb{Z}$.

The proof of Theorem 5.7 will require some technical preparation.

5.8 Definition. The *universal covering* of S^1 is the map $p\colon \mathbb{R}^1 \to S^1$ given by $p(s) = (\cos 2\pi s, \sin 2\pi s)$.

Geometrically p is the map that wraps $\mathbb{R}^1$ infinitely many times around the circle.

5.9 Note. For $y, y' \in \mathbb{R}$ we have $p(y) = p(y')$ if and only if $y' = y + n$ for some $n \in \mathbb{Z}$. As a consequence if $x_0 \in S^1$ and if $y_0 \in \mathbb{R}$ is a point such that $p(y_0) = x_0$ then $p^{-1}(x_0) = \{y_0 + n \mid n \in \mathbb{Z}\}$.

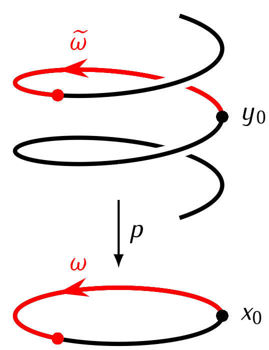


5.10 Definition. Let ω be a path in S^1. We say that a path $\widetilde{\omega}$ in $\mathbb{R}$ is a *lift* of ω if $p \circ \widetilde{\omega} = \omega$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {[0,1]} \arrow[ur, "\widetilde{\omega}"] \arrow[r, "\omega"'] & S^1 \end{tikzcd}$$

5.11 Proposition. *Let $p\colon \mathbb{R}^1 \to S^1$ be the universal covering of S^1, let $x_0 \in S^1$, and let $y_0 \in \mathbb{R}^1$ be a point such that $p(y_0) = x_0$.*

1) For any path $\omega\colon [0,1] \to S^1$ such that $\omega(0) = x_0$ there exists a lift $\widetilde{\omega}\colon [0,1] \to \mathbb{R}^1$ satisfying $\widetilde{\omega}(0) = y_0$. Moreover, such lift is unique.



2) Let $\omega, \tau\colon [0,1] \to S^1$ be paths such that $\omega(0) = \tau(0) = x_0$, $\omega(1) = \tau(1)$ and $\omega \simeq \tau$. If $\widetilde{\omega}$, $\widetilde{\tau}$ are lifts of ω, τ, respectively, such that $\widetilde{\omega}(0) = \widetilde{\tau}(0) = y_0$ then $\widetilde{\omega}(1) = \widetilde{\tau}(1)$ and $\widetilde{\omega} \simeq \widetilde{\tau}$.

We postpone the proof of Proposition 5.11 for now. We will get back to it in Chapter 17 where we will show that it is a special case of a more general statement. Meanwhile we will show how it can be used to obtain Theorem 5.7.

5.12 Definition. Let $x_0 \in S^1$ and $y_0 \in \mathbb{R}$ be points such that $p(y_0) = x_0$. Let ω be a loop in S^1 based at x_0 and let $\widetilde{\omega}$ be the unique lift of ω such that $\widetilde{\omega}(0) = y_0$. The *degree* of ω is the integer $\deg(\omega)$ such that $\widetilde{\omega}(1) = y_0 + \deg(\omega)$.

In other words $\deg(\omega) = \widetilde{\omega}(1) - \widetilde{\omega}(0)$.

5.13 Note. Notice that $\deg(\omega)$ does not depend on the choice of the point y_0, i.e. it does not depend on the choice of the lift of ω. Indeed, if $y_0' \in \mathbb{R}$ is another point satisfying $p(y_0') = x_0$ then $y_0' = y_0 + n$

for some $n \in \mathbb{Z}$. Also, if $\widetilde{\omega}$ is the lift of a loop ω with $\widetilde{\omega}(0) = y_0$ then the lift $\widetilde{\omega}'$ of ω with $\widetilde{\omega}'(0) = y_0'$ is given by $\widetilde{\omega}'(s) = \widetilde{\omega}(s) + n$. This gives

$$\widetilde{\omega}'(1) - \widetilde{\omega}'(0) = (\widetilde{\omega}(1) + n) - (\widetilde{\omega}(0) + n) = \widetilde{\omega}(1) - \widetilde{\omega}(0)$$

In addition, by part 2) of Proposition 5.11 $\deg(\omega)$ depends only on the homotopy class of ω, and thus we obtain a well-defined function

$$\deg\colon \pi_1(S^1, x_0) \to \mathbb{Z}$$

Proof of Theorem 5.7. Let $x_0 \in S^1$. We will show that the function $\deg\colon \pi_1(S^1, x_0) \to \mathbb{Z}$ is an isomorphism of groups.

First, we will show that deg is onto. Let $y_0 \in p^{-1}(x_0)$. Given $n \in \mathbb{Z}$ consider the path $\widetilde{\omega}_n\colon [0,1] \to \mathbb{R}$ given by $\widetilde{\omega}_n(s) = y_0 + ns$ and let $\omega_n = p \circ \widetilde{\omega}_n$. Since $\widetilde{\omega}_n$ is the lift of ω_n such that $\widetilde{\omega}_n(0) = y_0$ and since $\widetilde{\omega}_n(1) = y_0 + n$ we obtain $\deg[\omega_n] = n$.

Next, we will check that deg is a 1-1 function. Let $[\omega], [\tau] \in \pi_1(S^1, x_0)$ be elements such that $\deg[\omega] = \deg[\tau]$. We need to show that $[\omega] = [\tau]$. Let $\widetilde{\omega}$, $\widetilde{\tau}$ be the lifts of ω, τ, respectively, such that $\widetilde{\omega}(0) = \widetilde{\tau}(0) = y_0$. By assumption we get

$$\widetilde{\omega}(1) = y_0 + \deg[\omega] = y_0 + \deg[\tau] = \widetilde{\tau}(1)$$

Since $\mathbb{R}$ is a simply connected space using Proposition 5.6 we obtain that $\widetilde{\omega} \simeq \widetilde{\tau}$. Therefore

$$\omega = p \circ \widetilde{\omega} \simeq p \circ \widetilde{\tau} = \tau$$

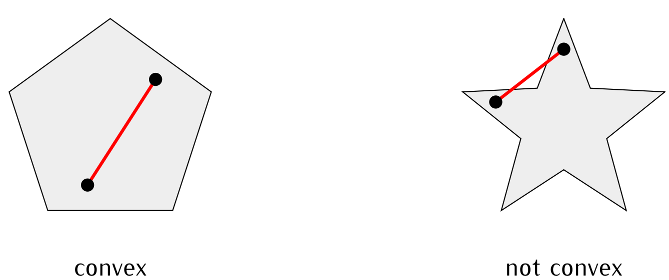
which gives $[\omega] = [\tau]$.

It remains to show that deg is a homomorphism of groups (exercise). $\square$

Exercises to Chapter 5

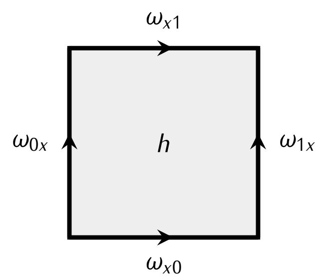
E5.1 Exercise. Prove Proposition 5.6.

E5.2 Exercise. a) A subspace $X \subseteq \mathbb{R}^n$ is *convex* is for any points $x_1, x_2 \in X$ the straight line segment joining these points is contained in X:



Show that if X is convex then it is simply connected.

b) Let Y be a topological space and let $h\colon [0,1]\times[0,1]\to Y$ be a continuous function. Consider paths ω_{x0}, ω_{x1}, ω_{0x}, ω_{1x} in Y which are defined by restricting h to the four edges of the square $[0,1]\times[0,1]$: $\omega_{x0}(s)=h(s,0)$, $\omega_{x1}(s)=h(s,1)$, $\omega_{0x}(s)=h(0,s)$ and $\omega_{1x}=h(1,s)$.



Show that the path $\omega_{x0}*\omega_{1x}$ is path homotopic to $\omega_{0x}*\omega_{x1}$.

E5.3 Exercise. Recall that the Peano curve is a continuous function $\tau\colon[0,1]\to[0,1]\times[0.1]$ which is onto, and such that $\tau(0)=(0,0)$ and $\tau(1)=(1,1)$. Let $\mathrm{pr}_1\colon[0,1]\times[0,1]$ be the projection onto the first factor, $\mathrm{pr}_1(s,t)=s$ and let $\omega\colon[0,1]\to S^1$ be standard the degree 1 loop, $\omega(s)=(\cos(2\pi s),\sin(2\pi s))$. The composition $\omega\circ\mathrm{pr}_1\circ\tau\colon[0,1]\to S^1$ is a loop with the basepoint at $x_0=(1,0)\in S^1$. Compute the degree of $\omega\circ\mathrm{pr}_1\circ\tau$.

E5.4 Exercise. Show that the degree function $\deg\colon \pi_1(S^1,x_0)\to\mathbb{Z}$ defined in Note 5.13 is a group homomorphism.

E5.5 Exercise. Let $\omega\colon[0,1]\to S^1$ be a loop based at $x_0\in S^1$. Assume that there exists a point $x\in S^1$ such that the set $\omega^{-1}(x)$ consists of n points. Show that $-n\leq\deg(\omega)\leq n$.