23 | Deck Transformations

23.1 Definition. Let $p\colon T \to X$ be a covering. A *deck transformation* of p is an isomorphism of coverings

$$\begin{array}{ccc} T & \xrightarrow[\cong]{f} & T \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & X & \end{array}$$

Deck transformations form a group under composition of isomorphisms. We will denote this group by $D(p)$. In this chapter we will compute the group $D(p)$ for a path connected covering p in terms of fundamental groups of X and T. Recall that in Chapter 22 we constructed a functor

$$\Lambda\colon \mathbf{PCov}(X) \to \mathbf{TSet}_{\pi_1(X,x_0)}$$

from the category of path connected coverings of a space X to the category of transitive $\pi_1(X, x_0)$-sets. We also showed (22.15) that if X is a connected and locally path connected space then this functor is a bijection of sets of morphisms. Since any functor preserves isomorphism, if $f : T_1 \to T_2$ is an isomorphism of coverings of X, then $\Lambda(f)$ is an isomorphism of $\pi_1(X, x_0)$-sets. The following fact implies that the converse is also true:

23.2 Lemma. *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor such that for any $c, c' \in \mathbf{C}$ the map the map $\mathrm{Mor}_{\mathbf{C}}(c, c') \to \mathrm{Mor}_{\mathbf{D}}(F(c), F(c'))$ given by $f \mapsto F(f)$ is a bijection. A morphism $f\colon c \to c'$ i in $\mathbf{C}$ is an isomorphism if and only if $F(f)\colon F(c) \to F(c')$ is an isomorphism.*

Proof. Exercise. □

As a consequence we obtain:

23.3 Corollary. *Let X be a connected and locally path connected space, $x_0 \in X$, and let $p\colon T \to X$ be a path connected covering. The group of deck transformations $D(p)$ is isomorphic to the group of $\pi_1(X, x_0)$-equivariant isomorphisms $p^{-1}(x_0) \to p^{-1}(x_0)$.*

Proof. Exercise. □

In view of Corollary 23.3 the problem of computing the group of deck transformations reduces to the problem of computing the group of G-equivariant isomorphisms of a G-set S. Denote this group by $\mathrm{Iso}_G(S)$.

23.4 Definition. Let G be a group, and let $H \subseteq G$ be a subgroup. The *normalizer* of H in G is the subgroup $N_G(H) \subseteq G$ defined by

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\}$$

23.5 Note. $N_G(H)$ is the largest subgroup of G that contains H as its normal subgroup. In particular H is a normal subgroup of G if and only if $N_G(H) = G$.

Recall that if S is a G-set and $s \in S$ then by G_s we denote the stabilizer of s.

23.6 Proposition. *Let G be a group, and let S is a transitive G-set. For any $s \in S$ there exists an isomorphism of groups*

$$\mathrm{Iso}_G(S) \cong N_G(G_s)/G_s$$

Proof. Let $f\colon S \to S$ be a G-equivariant isomorphism. Since the action of G on S is transitive we we have $f(s) = sg_f$ for some $g_f \in G$ (depending on s). We claim that $g_f \in N_G(G_s)$. Indeed, for any $h \in G_s$ we have

$$s(g_f h g_f^{-1}) = f(s)(hg_f^{-1}) = f(sh)g_f^{-1} = f(s)g_f^{-1} = s(g_f g_f^{-1}) = s$$

which shows that $g_f h g_f^{-1} \in G_s$.

Define a map

$$\varphi\colon \mathrm{Iso}_G(S) \to N_G(G_s)/G_s$$

by $\varphi(f) := g_f G_s$. To verify that φ is well defined we need to check that if $\bar{g}_f \in G$ is another element such that $f(s) = s\bar{g}_f$ then $g_f G_s = \bar{g}_f G_s$. Since $sg_f = f(s) = s\bar{g}_f$ we get $s = s\bar{g}_f g_f^{-1}$ which gives $\bar{g}_f g_f^{-1} \in G_s$. By the observation above $g_f \in N_G(G_s)$, so $(\bar{g}_f g_f^{-1})g_f = g_f h$ for some $h \in G_s$. This gives:

$$\bar{g}_f G_s = \bar{g}_f g_f^{-1} g_f G_s = g_f h G_s = g_f G_s$$

Next, we claim that φ is a group homomorphism. Indeed, if $f, f' \in \mathrm{Iso}_G(S)$, $f(s) = sg_f$, $f'(s) = sg_{f'}$ then

$$f' \circ f(s) = f'(sg_f) = f'(s)g_f = sg_{f'}g_f$$

and so $\varphi(f' \circ f) = (g_{f'}g_f)G_s = \varphi(f') \cdot \varphi(f)$. It remains to show that φ is an isomorphism (exercise). □

23.7 Proposition. *Let X be a connected and locally path connected space, and let $x_0 \in X$. For a path connected covering $p\colon T \to X$ and $\tilde{x} \in p^{-1}(x_0)$ there exists an isomorphism of groups:*

$$D(p) \cong N_{\pi_1(X,x_0)}(p_*(\pi_1(T,\tilde{x})))/p_*(\pi_1(T,\tilde{x}))$$

23.8 Note. Recall that a covering $p\colon T \to X$ is regular if $p_*(\pi_1(T,\tilde{x}))$ is a normal subgroup of $\pi_1(X,x_0)$. In such case the isomorphism in Proposition 23.7 gives

$$D(p) \cong \pi_1(X,x_0)/p_*(\pi_1(T,\tilde{x}))$$

In particular, for the universal covering $\tilde{p}\colon \widetilde{X} \to X$ we obtain $D(\tilde{p}) \cong \pi_1(X,x_0)$.

Exercises to Chapter 23

E23.1 Exercise. For a function $f\colon X \to X$ by $\mathrm{Fix}(f)$ we will denote the set of fixed points of f:

$$\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}$$

Let X be a connected and locally path connected space, let $\widetilde{p}\colon \widetilde{X} \to X$ be the universal covering of X, and let $f\colon X \to X$ be a map. We will say that a map $\widetilde{f}\colon \widetilde{X} \to \widetilde{X}$ is a lift of f if the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{X} \\ {\scriptstyle \widetilde{p}}\downarrow & & \downarrow{\scriptstyle \widetilde{p}} \\ X & \xrightarrow[f]{} & X \end{array}$$

Let S denote the set of all lifts of f.

a) Show that $\mathrm{Fix}(f) = \bigcup_{\widetilde{f}\in S} \widetilde{p}(\mathrm{Fix}(\widetilde{f}))$.

b) Let $\widetilde{f}_1, \widetilde{f}_2 \in S$. Show that the following conditions are equivalent:

(i) $\widetilde{p}(\mathrm{Fix}(\widetilde{f}_1)) \cap \widetilde{p}(\mathrm{Fix}(\widetilde{f}_2)) \neq \varnothing$

(ii) There exists a deck transformation $g\colon \widetilde{X} \to \widetilde{X}$ such that $\widetilde{f}_2 = g\widetilde{f}_1 g^{-1}$

(iii) $\widetilde{p}(\mathrm{Fix}(\widetilde{f}_1)) = \widetilde{p}(\mathrm{Fix}(\widetilde{f}_2))$

c) Let $f\colon (S^1, x_0) \to (S^1, x_0)$ be a map such that the homomorphism $f_*\colon \pi_1(S^1,x_0) \to \pi_1(S^1,x_0)$ is given by $f_*([\omega]) = n\cdot[\omega]$ for some $n \in \mathbb{Z}$. Show that $\mathrm{Fix}(f)$ consists of at least $|n-1|$ points.