17 | Covering Spaces

All computations of non-trivial fundamental groups we have seen use the fact that the group $\pi_1(S^1)$ is isomorphic to the group of integers. The proof of this fact, however, is still incomplete since it relies on the path lifting property of the universal covering of S^1 (Proposition 5.11) that we left without justification. Our next goal is to fill this gap. In this chapter we define the notion of a *covering* of a space and we show that the path lifting property holds for any covering. Since the universal covering of S^1 is an example of a covering this will give in particular a proof of Proposition 5.11.

17.1 Definition. A map $p\colon T \to X$ is a *covering* of X if for every point $x \in X$ there exists an open neighborhood $U_x \subseteq X$ and a homeomorphism $h_{U_x}\colon p^{-1}(U_x) \to U_x \times D_x$ where D_x is some discrete space, such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U_x) \arrow[rr, "h_{U_x}"] \arrow[dr, "p"'] & & U_x \times D_x \arrow[dl, "\mathrm{pr}_1"] \\ & U_x & \end{tikzcd}$$

Here $\mathrm{pr}_1\colon U_x \times D_x \to U_x$ is the projection map $\mathrm{pr}_1(y, d) = y$.

17.2 Here is some terminology and a few comments related to the notion of a covering.

1) Given a covering $p\colon T \to X$ we will say that T is the *total space* of p and that X is the *base space*.

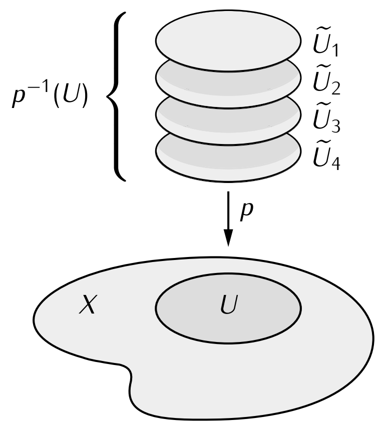
2) For $x \in X$ we have $p^{-1}(x) \cong \{x\} \times D_x \cong D_x$ which means that $p^{-1}(x)$ is a discrete space. We call $p^{-1}(x)$ the *fiber* of the covering p over the point x.

3) In general for $x, x' \in X$ the fibers $p^{-1}(x)$ and $p^{-1}(x')$ may have different numbers of points, and so they don't need to be homeomorphic. However, if the space X is connected then $p^{-1}(x) \cong p^{-1}(x')$ for all $x, x' \in X$ (exercise).

4) If $p^{-1}(x)$ consists of n points for all $x \in X$ then we say that p is an n-fold covering of X.

5) If $U \subseteq X$ is an open set such that for some discrete space D there exists a homeomorphism $h_U\colon p^{-1}(U) \to U \times D$ satisfying $\mathrm{pr}_1 h_U = p$ then we say that the set U is *evenly covered*. Definition of a covering can be rephrased by saying that every point $x \in X$ has an open neighborhood which is evenly covered.

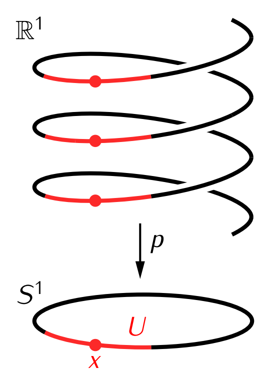
6) If $U \subseteq X$ is an evenly covered set and $h_U\colon p^{-1}(U) \to U \times D$ is a homeomorphism then for $d \in D$ we will say that the set $\widetilde{U}_d = h_U^{-1}(U \times \{d\}) \subseteq p^{-1}(U)$ is a *slice* over U. The set $p^{-1}(U)$ is then a disjoint union of slices:



Moreover, for each slice $\widetilde{U}_d$ the map $p|_{\widetilde{U}_d}\colon \widetilde{U}_d \to U$ is a homeomorphism.

17.3 Example. Let D be a discrete space. The projection map $\mathrm{pr}_1\colon X \times D \to X$ is a covering of X. In this case the whole space X is evenly covered. We say that $\mathrm{pr}_1\colon X \times D \to X$ is a *trivial covering* of X.

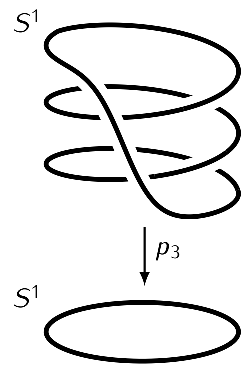
17.4 Example. Recall that the universal covering of S^1 is the map $p\colon \mathbb{R}^1 \to S^1$ given by $p(s) = (\cos 2\pi s, \sin 2\pi s)$. If $U \subseteq S^1$ is any open set such that $U \neq S^1$, then $p^{-1}(U)$ is evenly covered and $p^{-1}(U) \cong U \times \mathbb{Z}$ (exercise).



17.5 Example. Consider S^1 as a subset of the complex plane:

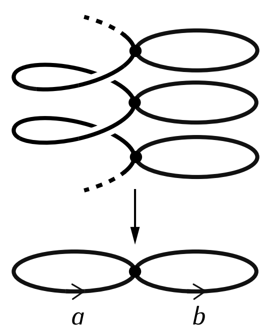
$$S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$$

For $n = 1, 2, \dots$ the map $p_n : S^1 \to S^1$ given by $p_n(z) = z^n$ is an n-fold covering of S^1. Similarly as in the case of the universal covering of S^1 any open set $U \subseteq S^1$ such that $U \neq S^1$ is evenly covered by p_n (exercise).

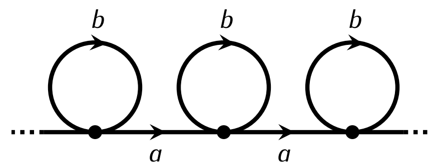


17.6 Example. If $p_1 : T_1 \to X_1$ and $p_2 : T_2 \to X_2$ are coverings then the map $p_1 \times p_2 : T_1 \times T_2 \to X_1 \times X_2$ is also a covering (exercise). For example, starting with the universal covering $p : \mathbb{R}^1 \to S^1$ of the circle we obtain a covering $p \times p : \mathbb{R}^1 \times \mathbb{R}^1 \to S^1 \times S^1$ of the torus.

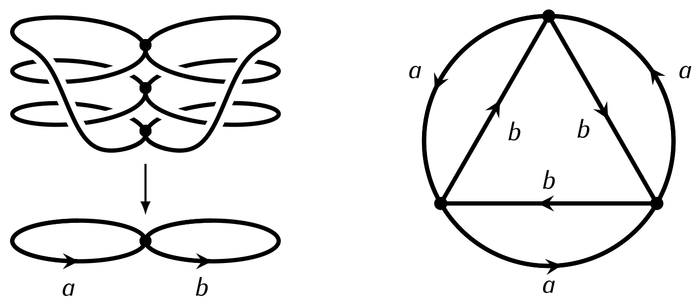
17.7 Example. Using the coverings of S^1 described above we can construct many coverings of $S^1 \vee S^1$. For example, here is a covering obtained by combining the universal covering over one copy of S^1 and a trivial covering over the second copy:

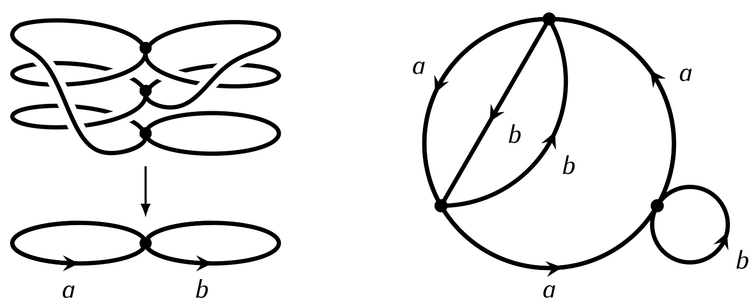


Coverings like this are easier to represent graphically if we untangle the total space and indicate which of its parts are being mapped to which copy of S^1. For the covering depicted above this gives:



Here are two different 3-fold coverings of $S^1 \vee S^1$:





17.8 Definition. If $p\colon T \to X$ is a covering and $f\colon Y \to X$ is a map then a *lift* of f is a map $\tilde{f}\colon Y \to T$ such that the following diagram commutes:

$$\begin{tikzcd} & T \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

The following fact describes one of the main properties of coverings:

17.9 Theorem (Homotopy Lifting Property). *Let $p\colon T \to X$ be a covering. Let $F\colon Y \times [0,1] \to X$ and $\tilde{f}\colon Y \times \{0\} \to T$ be functions satisfying $p\tilde{f} = F|_{Y\times\{0\}}$. There exists a function $\widetilde{F}\colon Y \times [0,1] \to T$ such that $p\widetilde{F} = F$ and $\widetilde{F}|_{Y\times\{0\}} = \tilde{f}$:*

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{f}"] \arrow[d, hook] & T \arrow[d, "p"] \\ Y \times [0,1] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & X \end{tikzcd} \tag{$*$}$$

Moreover, such function $\widetilde{F}$ is unique.

Before we get to the proof of this theorem we will show that it implies that any covering has path lifting properties analogous to the ones described in Proposition 5.11 for the universal covering of S^1:

17.10 Corollary. *Let $p\colon T \to X$ be a covering. Let $x_0 \in X$, and let $\tilde{x}_0 \in T$ be a point such that $p(\tilde{x}_0) = x_0$.*

1) For any path $\omega\colon [0,1] \to X$ such that $\omega(0) = x_0$ there exists a lift $\widetilde{\omega}\colon [0,1] \to T$ satisfying $\widetilde{\omega}(0) = \tilde{x}_0$. Moreover, such lift is unique.

2) Let $\omega, \tau\colon [0,1] \to X$ be paths such that $\omega(0) = \tau(0) = x_0$, $\omega(1) = \tau(1)$ and $\omega \simeq \tau$. If $\widetilde{\omega}$, $\widetilde{\tau}$ are lifts of ω, τ, respectively, such that $\widetilde{\omega}(0) = \widetilde{\tau}(0) = \tilde{x}_0$ then $\widetilde{\omega}(1) = \widetilde{\tau}(1)$ and $\widetilde{\omega} \simeq \widetilde{\tau}$.

Proof. For part 1) let $Y = \{*\}$ be the space consisting of one point. We can consider the path ω as a map $\omega\colon \{*\} \times [0,1] \to X$. Denote by $c_{\tilde{x}_0}\colon \{*\} \times \{0\} \to T$ the map given by $c_{\tilde{x}_0}(*, 0) = \tilde{x}_0$. We have a commutative diagram:

$$\begin{tikzcd} \{*\} \times \{0\} \arrow[r, "c_{\tilde{x}_0}"] \arrow[d, hook] & T \arrow[d, "p"] \\ \{*\} \times [0,1] \arrow[r, "\omega"'] & X \end{tikzcd}$$

By Theorem 17.9 there exists a unique map $\widetilde{\omega}\colon \{*\} \times [0,1] \to T$ which gives the desired lift of ω.

Part 2) is an exercise.

□

Proof of Theorem 17.9 will use a couple of lemmas. The first of them if of interest of its own right:

17.11 Lemma. *Let $p\colon T \to X$ be a covering, and let $\tilde{f}_1, \tilde{f}_2\colon Y \to T$ be two lifts of a map $f\colon Y \to X$. If Y is a connected space and there exists $y_0 \in Y$ such that $\tilde{f}_1(y_0) = \tilde{f}_2(y_0)$ then $\tilde{f}_1(y) = \tilde{f}_2(y)$ for all*

$y \in Y$.

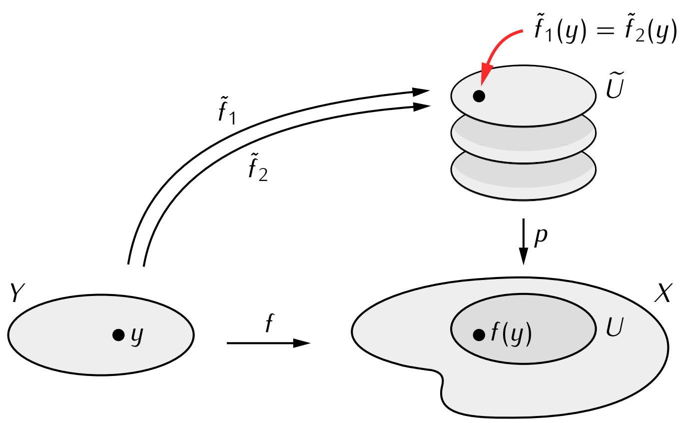
$$\begin{tikzcd} & T \arrow[d, "p"] \\ Y \arrow[ur, shift left, "\tilde{f}_1"] \arrow[ur, shift right, "\tilde{f}_2"'] \arrow[r, "f"'] & X \end{tikzcd}$$

Proof. Let $Y_e, Y_n \subseteq X$ be sets defined by

$$Y_e = \{y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y)\} \quad \text{and} \quad Y_n = \{y \in Y \mid \tilde{f}_1(y) \neq \tilde{f}_2(y)\}$$

Notice that $Y_e \cup Y_n = Y$ and $Y_e \cap Y_n = \varnothing$. Notice also that $Y_e \neq \varnothing$ since $y_0 \in Y_e$. It will be enough to show that Y_e and Y_n are open in Y. By connectedness of Y this will imply that $Y_e = Y$.

To see that Y_e is open take $y \in Y_e$. It will suffice to show that there exists an open set $V \subseteq Y$ such that $y \in V$ and $V \subseteq Y_e$. Let $U \subseteq X$ be an evenly covered open neighborhood of $f(y)$ and let $\widetilde{U}$ be a slice over U such that $\tilde{f}_1(y) = \tilde{f}_2(y) \in \widetilde{U}$.



Take $V = \tilde{f}_1^{-1}(\widetilde{U}) \cap \tilde{f}_2^{-1}(\widetilde{U})$. The set V is an open neighborhood of y. Also, for $y' \in V$ we have

$$p|_{\widetilde{U}} \circ \tilde{f}_1(y') = f(y') = p|_{\widetilde{U}} \circ \tilde{f}_2(y')$$

Since $p|_{\widetilde{U}} \colon \widetilde{U} \to U$ is a homeomorphism this gives $\tilde{f}_1(y') = \tilde{f}_2(y')$, and so $y' \in Y_e$.

Openness of Y_n can be verified in a similar way (exercise). □

The next lemma is a special case of Theorem 17.9:

17.12 Lemma. *Let $p \colon T \to X$ be a covering. Let $F \colon Y \times [a, b] \to X$ and $\tilde{f} \colon Y \times \{a\} \to T$ be functions satisfying $p\tilde{f} = F|_{Y \times \{a\}}$. Assume also that $F(Y \times [a, b]) \subseteq U$ where $U \subseteq X$ is an evenly covered open* *set. There exists a function $\widetilde{F}\colon Y \times [a, b] \to T$ such that $p\widetilde{F} = F$ and $\widetilde{F}|_{Y\times\{a\}} = \tilde{f}$:*

$$\begin{tikzcd} Y \times \{a\} \arrow[r, "\tilde{f}"] \arrow[d, hook] & T \arrow[d, "p"] \\ Y \times [a,b] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & X \end{tikzcd}$$

Proof. Let $\{\widetilde{U}_i\}_{i\in I}$ be the set of slices of p over the set U, and let $V_i = \{y \in Y \mid \tilde{f}(y, a) \in \widetilde{U}_i\}$. The sets $V_i \times [a, b]$ form an open cover of $Y \times [a, b]$. Since for each $i \in I$ the map $p|_{\widetilde{U}_i}\colon \widetilde{U}_i \to U$ is a homeomorphism we can define

$$\widetilde{F}|_{V_i\times[a,b]} = (p|_{\widetilde{U}_i})^{-1} \circ F|_{V_i\times[a,b]}$$

Since for $i \neq j$ we have $V_i \cap V_j = \varnothing$ this gives a well defined continuous function $\widetilde{F}\colon Y \times [a, b] \to T$ □

Proof of Theorem 17.9. We will show first that if the map $\widetilde{F}$ exists then it must be unique. Assume that for $i = 1, 2$ we have a map $\widetilde{F}_i$ such that the following diagram commutes:

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{f}"] \arrow[d, hook] & T \arrow[d, "p"] \\ Y \times [0,1] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}_i"] & X \end{tikzcd}$$

Take $y \in Y$. Notice that the set $\{y\} \times [0, 1] \subseteq Y \times [0, 1]$ is connected, for $i = 1, 2$ the map $\widetilde{F}_i|_{\{y\}\times[0,1]}$ is a lift of $F|_{\{y\}\times[0,1]}$, and $\widetilde{F}_1|_{\{y\}\times[0,1]}(y, 0) = \tilde{f}(y, 0) = \widetilde{F}_2|_{\{y\}\times[0,1]}(y, 0)$. By Lemma 17.11 we obtain that $\widetilde{F}_1|_{\{y\}\times[0,1]} = \widetilde{F}_2|_{\{y\}\times[0,1]}$ for each $y \in Y$. Therefore $\widetilde{F}_1 = \widetilde{F}_2$.

It remains to show that a map $\widetilde{F}$ in the diagram (∗) exists. Let $y \in Y$. We will construct first a map $\widetilde{F}_y\colon V_y \times [0, 1] \to T$ where $V_y \subseteq Y$ is some open neighborhood of y, such that the following diagram commutes:

$$\begin{tikzcd} V_y \times \{0\} \arrow[r, "\tilde{f}|_{V_y\times\{0\}}"] \arrow[d, hook] & T \arrow[d, "p"] \\ V_y \times [0,1] \arrow[r, "F|_{V_y\times[0,1]}"'] \arrow[ur, dashed, "\widetilde{F}_y"] & X \end{tikzcd} \tag{$**$}$$

The construction of $\widetilde{F}_y$ proceeds as follows. Using compactness of the interval $[0, 1]$ we can find an open neighborhood $V_y \subseteq Y$ of y and numbers $0 = t_0 < t_1 < \ldots < t_n = 1$ such that for each $i = 1, \ldots, n$

we have $F(V_y \times [t_{i-1}, t_i]) \subseteq U_i$ where $U_i \subseteq X$ is an evenly covered open set. Using Lemma 17.12 we obtain a map $\widetilde{F}_{y,1} \colon V_y \times [t_0, t_1] \to T$ that gives a commutative diagram

$$\begin{tikzcd}
V_y \times \{0\} \arrow[r, "\tilde{f}|_{V_y \times \{0\}}"] \arrow[d, hook] & T \arrow[d, "p"] \\
V_y \times [t_0, t_1] \arrow[r, "F|_{V_y \times [t_0, t_1]}"'] \arrow[ur, dashed, "\widetilde{F}_{y,1}"] & X
\end{tikzcd}$$

Next, using Lemma 17.12 again we obtain a map $\widetilde{F}_{y,2} \colon V_y \times [t_1, t_2] \to T$ that fits into the commutative diagram

$$\begin{tikzcd}
V_y \times \{t_1\} \arrow[r, "\widetilde{F}_{y,1}|_{V_y \times \{t_1\}}"] \arrow[d, hook] & T \arrow[d, "p"] \\
V_y \times [t_1, t_2] \arrow[r, "F|_{V_y \times [t_0, t_1]}"'] \arrow[ur, dashed, "\widetilde{F}_{y,2}"] & X
\end{tikzcd}$$

Arguing inductively we obtain in this way for each $k = 1, \dots, n$ a map $\widetilde{F}_{y,k} \colon V_y \times [t_{k-1}, t_k] \to T$. By construction $\widetilde{F}_{y,k}|_{V_y \times \{t_k\}} = \widetilde{F}_{y,k+1}|_{V_y \times \{t_k\}}$ for all $k = 1, \dots, n-1$, so these maps taken together define a map $\widetilde{F}_y \colon V_y \times [0,1] \to T$ that fits into the commutative diagram (∗∗).

Next, we would like to take $\widetilde{F} \colon Y \times [0,1] \to T$ to be the map such that $\widetilde{F}|_{V_y \times [0,1]} = \widetilde{F}_y$ for each $y \in Y$. In order to verify that such map is well defined we need to show that if $z \in V_y \cap V_{y'}$ for some $y, y' \in Y$ then $F_y|_{\{z\}\times[0,1]} = F_{y'}|_{\{z\}\times[0,1]}$. This however holds by Lemma 17.11 since both $F_y|_{\{z\}\times[0,1]}$ and $F_{y'}|_{\{z\}\times[0,1]}$ are lifts of the map $F|_{\{z\}\times[0,1]}$, the set $\{z\} \times [0,1]$ is connected, and $F_y|_{\{z\}\times[0,1]}(z,0) = \tilde{f}(z,0) = F_{y'}|_{\{z\}\times[0,1]}(z,0)$. □

Exercises to Chapter 17

E17.1 Exercise. a) Let $p_i \colon T_i \to X_i$ be a covering for $i = 1, 2$. Show that the map $p_1 \times p_2 \colon T_1 \times T_2 \to X_1 \times X_2$ is a covering.

b) Give an example showing that if $p_i \colon T_i \to X_i$ is a covering for $i = 1, 2, \dots$ then the map

$$\prod_{i=1}^{\infty} p_i \colon \prod_{i=1}^{\infty} T_i \to \prod_{i=1}^{\infty} X_i$$

need not be a covering. Justify your answer.

E17.2 Exercise. Let $p\colon T \to X$ be a covering and let $U \subseteq T$ be an open set. Show that the set $p(U)$ is open in X.

E17.3 Exercise. Let $p\colon T \to X$ be a covering such that X is a connected space. Show that for any points $x, x' \in X$ there exists a bijection between $p^{-1}(x)$ and $p^{-1}(x')$.

E17.4 Exercise. Prove part 2) of Corollary 17.10.