1 | Some Motivation

The main idea behind algebraic topology is that, in order to solve problems involving topological spaces one can try to translate them into problems about algebraic objects (groups, vectors spaces, rings, modules etc.) and then solve the resulting algebraic problems. The translation between topology and algebra is achieved by constructing assignments of the form:

$$\begin{aligned} \text{topological spaces} &\longmapsto \text{groups (rings, modules, \dots)} \\ \text{continuous functions} &\longmapsto \text{homomorphisms of groups (or rings, modules, \dots)} \end{aligned}$$

For example, one of the main objectives of these notes is to study the assignment that associates to each space X a group $\pi_1(X)$ which is called the *fundamental group* of X [1]. Let S^1 denote the unit circle and D^2 the closed unit disc:

$$\begin{aligned} S^1 =&\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\} \\ D^2 =&\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\} \end{aligned}$$

We will see that $\pi_1(S^1) \cong \mathbb{Z}$ and that $\pi_1(D^2)$ is the trivial group. Since homeomorphic spaces have isomorphic fundamental groups an immediate consequence is that $S^1 \not\cong D^2$. This is one typical application of algebraic invariants appearing in algebraic topology: they provide a tool for detecting if topological spaces are homeomorphic or not. However, these invariants can be also used to study more subtle relationships between spaces. Recall, for example, that if X is a topological space then we say that a subspace $A \subseteq X$ is a retract of X if there exists a continuous function $r\colon X \to A$ such that $r(x) = x$ for all $x \in A$.

1.1 Example. Let $\mathbf{0} = (0, 0)$ be the center of the disc D^2. Define $r\colon D^2 \smallsetminus \{\mathbf{0}\} \to S^1$ by

$$r(x) = \frac{x}{||x||}$$

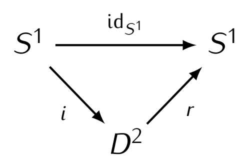
[1] Technically $\pi_1(X)$ depends not only on the space X but also on the choice of a basepoint $x_0 \in X$, but we will disregard this for a moment.

where, for $x = (x_1, x_2)$ we set $\|x\| = \sqrt{x_1^2 + x_2^2}$. Since $r(x) = x$ for all $x \in S^1$ this shows that S^1 is a retract of $D^2 \setminus \{\mathbf{0}\}$.

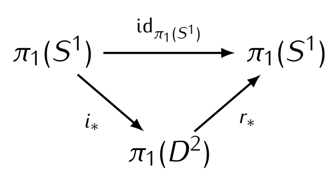
On the other hand we have:

1.2 Proposition. *The circle S^1 is not a retract of D^2.*

Idea of a proof. We argue by contradiction. Let $i\colon S^1 \to D^2$ be the inclusion map. If S^1 is a retract of D^2, then there exists a map $r\colon D^2 \to S^1$ such that $ri = \mathrm{id}_{S^1}$, i.e. such that the following diagram commutes:



The construction of the fundamental group associates to any continuous function of topological spaces $f\colon X \to Y$, a homomorphism of groups $f_*\colon \pi_1(X) \to \pi_1(Y)$ in a way that preserves compositions (i.e. $(fg)_* = f_* g_*$) and maps identity functions to identity group homomorphisms: $\mathrm{id}_{X*} = \mathrm{id}_{\pi_1(X)}$. As a result, the above commutative diagram of topological spaces gives a commutative diagram of groups:



This implies, in particular, that r_* is onto, which is impossible since $\pi_1(D^2)$ is the trivial group and $\pi_1(S^1)$ is non-trivial. $\square$