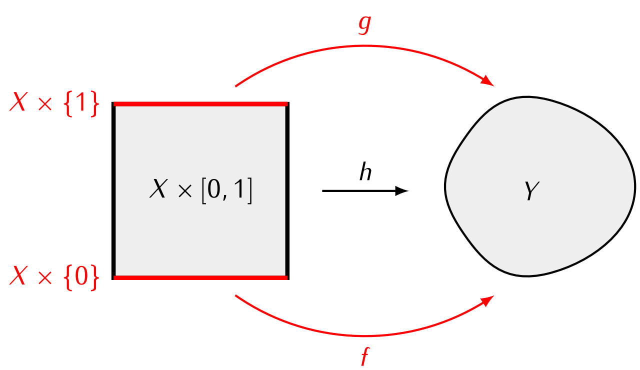
8 | Homotopy Invariance

8.1 Definition. Let $f, g\colon X \to Y$ be continuous functions. A *homotopy* between f and g is a continuous function $h\colon X \times [0,1] \to Y$ such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$:



If such homotopy exists then we say that the functions f and g are *homotopic* and we write $f \simeq g$. We will also write $h\colon f \simeq g$ to indicate that h is a homotopy between f and g.

8.4 Definition. Let X be a space and let $A \subseteq X$. If $f, g\colon X \to Y$ are functions such that $f|_A = g|_A$ then we say that f and g are *homotopic relative to* A if there exists a homotopy $h\colon X \times [0,1] \to Y$ such that $h_0 = f$, $h_1 = g$ and $h_t|_A = f|_A = g|_A$ for all $t \in [0,1]$. In such case we write $f \simeq g$ (rel A).

8.6 Definition. A map $f\colon X \to Y$ is a *homotopy equivalence* if there exists a map $g\colon Y \to X$ such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$. If such maps exist we say that the spaces X and Y are *homotopy equivalent* and we write $X \simeq Y$.

8.10 Definition. If X is a space such that $X \simeq \{*\}$ then we say that X is a *contractible space*.

8.11 Proposition. *Let X be a topological space. The following conditions are equivalent:*

1) *X is contractible;*
2) *the identify map id_X is homotopic to a constant map;*
3) *for each space Y and any maps $f, g\colon Y \to X$ we have $f \simeq g$.*

Proof. Exercise. □

8.12 Definition. A subspace $A \subseteq X$ is a *deformation retract* of a space X if there exists a homotopy $h\colon X \times [0,1] \to X$ such that

1) $h_0 = \mathrm{id}_X$
2) $h_t|_A = \mathrm{id}_A$ for all $t \in [0,1]$
3) $h_1(x) \in A$ for all $x \in X$

In such case we say that h is a *deformation retraction* of X onto A.

8.13 Proposition. *If $A \subseteq X$ is a deformation retract of X then $A \simeq X$.*

8.15 Definition. Mapping cylinder and mapping cone.

8.16 Proposition. *For any map $f\colon X \to Y$ we have $M_f \simeq Y$.*

Proof. Exercise. □

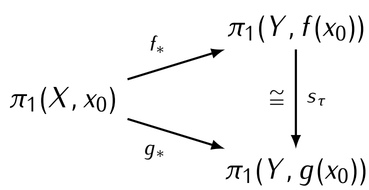
8.17 Proposition. *Let $f, g\colon X \to Y$ be continuous functions. If $f \simeq g$ then $C_f \simeq C_g$.*

Proof. Exercise. □

8.18 Example.

8.19 Proposition. *If $f, g\colon (X, x_0) \to (Y, y_0)$ are maps of pointed spaces such that $f \simeq g$ (rel $\{x_0\}$) then $f_* = g_*$.*

8.20 Proposition. *Let $f, g\colon X \to Y$ be homotopic maps and let $h\colon f \simeq g$. For $x_0 \in X$ let τ be the path in Y given by $\tau(t) = h(x_0, t)$. The following diagram commutes:*



Proof. Exercise. □

8.21 Corollary. *If $f, g\colon X \to Y$ are maps such that $f \simeq g$ then the homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism (or is trivial or is 1-1 or onto) if and only if the homomorphism $g_*\colon \pi_1(X, x_0) \to \pi_1(Y, g(x_0))$ has the same property.*

8.22 Proposition. *If $f\colon X \to Y$ is a homotopy equivalence then for any $x_0 \in X$ the homomorphism $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.*

8.23 Corollary. *If X, Y are path connected spaces and $X \simeq Y$ then $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$ for any $x_0 \in X$, $y_0 \in Y$.*