23 | Deck Transformations

23.1 Definition. Let $p\colon T \to X$ be a covering. A *deck transformation* of p is an isomorphism of coverings

$$\begin{array}{ccc} T & \xrightarrow[\cong]{f} & T \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p} & \\ & X & \end{array}$$

23.2 Lemma. *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a functor such that for any $c, c' \in \mathbf{C}$ the map the map $\mathrm{Mor}_{\mathbf{C}}(c, c') \to \mathrm{Mor}_{\mathbf{D}}(F(c), F(c'))$ given by $f \mapsto F(f)$ is a bijection. A morphism $f\colon c \to c'$ i in $\mathbf{C}$ is an isomorphism if and only if $F(f)\colon F(c) \to F(c')$ is an isomorphism.*

Proof. Exercise. □

23.3 Corollary. *Let X be a connected and locally path connected space, $x_0 \in X$, and let $p\colon T \to X$ be a path connected covering. The group of deck transformations $D(p)$ is isomorphic to the group of $\pi_1(X, x_0)$-equivariant isomorphisms $p^{-1}(x_0) \to p^{-1}(x_0)$.*

Proof. Exercise. □

23.4 Definition. Let G be a group, and let $H \subseteq G$ be a subgroup. The *normalizer* of H in G is the subgroup $N_G(H) \subseteq G$ defined by

$$N_G(H) = \{g \in G \mid gHg^{-1} = H\}$$

23.6 Proposition. *Let G be a group, and let S is a transitive G-set. For any $s \in S$ there exists an isomorphism of groups*

$$\mathrm{Iso}_G(S) \cong N_G(G_s)/G_s$$

23.7 Proposition. *Let X be a connected and locally path connected space, and let $x_0 \in X$. For a path connected covering $p\colon T \to X$ and $\tilde{x} \in p^{-1}(x_0)$ there exists an isomorphism of groups:*

$$D(p) \cong N_{\pi_1(X,x_0)}(p_*(\pi_1(T,\tilde{x})))/p_*(\pi_1(T,\tilde{x}))$$