22 Coverings and Group Actions

22.1 Definition. Let G be a group and S be a set. We say that G acts on X on the right if there exists a function

$$\mu: S \times G \to S$$

such that

- (i) $\mu(s, e) = s$ for any $s \in S$, where $e \in G$ is the trivial element;
- (ii) $\mu(\mu(s,g),h) = \mu(s,gh)$ for all $s \in S$, $h,g \in G$.

| 22.4 Definition. | We say t | hat a group | G acts | on set | S transitively | if for | any s, s' | $\in S$ t | here | exists |
|---------------------|----------|-------------|--------|--------|----------------|--------|-------------|-----------|------|--------|
| $g \in G$ such that | sg = s'. | | | | | | | | | |

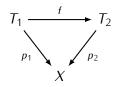
22.5 Proposition. Let $p: T \to X$ be a covering, and let $x_0 \in X$. If T is path connected then the monodromy action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ is transitive.

Proof. Exercise.

22.6 Definition. Let G be a group and let S, S' be G-sets. A function $f: S \to S'$ is G-equivariant if f(sg) = f(s)g for all $s \in S$ and $g \in G$.

22.7 Note. G-sets and G-equivariant functions form a category which we will denote by \mathbf{Set}_G .

22.8 Proposition. Let X be a space, and let



be a map of coverings. For any $x_0 \in X$ the induced map of fibers $f: p_1^{-1}(x_0) \to p_2^{-1}(x_0)$ is $\pi_1(X, x_0)$ -equivariant.

Proof. Exercise. □

| 22.9 Corollary. Let X be a space and let $x_0 \in X$. The assignment which associates to each path |
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| connected covering $p: T \to X$ the $\pi_1(X, x_0)$ -set $p^{-1}(x_0)$ and to each map of coverings the map of fibers |
| defines a functor |

$$\Lambda \colon \mathbf{Cov}(X) \to \mathbf{Set}_{\pi_1(X,x_0)}$$

Proof. Exercise.

22.10 Theorem. Let X be a connected, locally path connected, and semi-locally simply connected space, and let $x_0 \in X$. The functor

$$\Lambda \colon \mathsf{PCov}(X) \to \mathsf{TSet}_{\pi_1(X,x_0)}$$

is an equivalence of categories.

Outline of proof of Theorem 22.10:

- 1) We need to show that any set with a transitive action of the group $\pi_1(X, x_0)$ is isomorphic to a $\pi_1(X, x_0)$ -set $\Lambda(p \colon T \to X) = p^{-1}(x_0)$ for some path connected covering p.
- 2) We also need to show that maps of path connected coverings of X are in a bijective correspondence with $\pi_1(X, x_0)$ -equivariant maps of their fibers.

22.11 Proposition. Let X be a connected, locally path connected, and semi-locally simply connected space and let $x_0 \in X$. The map

given by $\Lambda(p: T \to X) = p^{-1}(x_0)$ is a bijection.

22.12 Definition. Let G be a group, and S be a G-set. The *stabilizer* of en element $s \in S$ is the subgroup $G_s \subseteq G$ given by:

$$G_s = \{g \in G \mid sg = s\}$$

22.13 Proposition. Let $p: T \to X$ be a covering, and let $x_0 \in X$. The stabilizer of an element $\tilde{x} \in p^{-1}(x_0)$ under the monodromy action is the subgroup $p_*(\pi_1(T, \tilde{x})) \subseteq \pi_1(X, x_0)$.

Proof. Exercise.

22.14 Lemma. Let G be a group.

- 1) If G acts transitively on a set S and $s, s' \in S$ then the stabilizers G_s and $G_{s'}$ are conjugate subgroups of the group G.
- 2) Let S be a set with an action of G and let $s \in S$. The assignment $S \mapsto G_s$ defines a bijective correspondence:

$$\Phi \colon \left(\begin{array}{c} \textit{isomorphism classes} \\ \textit{of sets with a transitive} \\ \textit{action of } G \end{array} \right) \quad \longrightarrow \quad \left(\begin{array}{c} \textit{conjugacy classes} \\ \textit{of subgroups} \\ \textit{of } G \end{array} \right)$$

22.15 Proposition. Let X be a connected and locally path connected space, and let $x_0 \in X$. For any path connected coverings $p_i \colon T_i \to X$, i = 1, 2 the assignment

$$\Lambda: \left(\begin{array}{c} maps \ of \ coverings \\ T_1 \to T_2 \end{array} \right) \longrightarrow \left(\begin{array}{c} \pi_1(X, x_0) \text{-equivariant maps} \\ p_1^{-1}(x_0) \to p_2^{-1}(x_0) \end{array} \right)$$

is a bijection.

22.16 Lemma. Let S, T be sets with a transitive action of a group G, and let $s_0 \in S$, $t_0 \in T$. A G-equivariant map $f: S \to T$ such that $f(s_0) = t_0$ exists if and only if $G_{s_0} \subseteq G_{t_0}$. Moreover, if such map exists then it is unique.