21 | Equivalences of Categories

21.1 Theorem. *Let X be a connected, locally path connected, and semi-locally simply connected space, and let $x_0 \in X$. The map*

$$\Omega\colon \left(\begin{array}{c}\textit{isomorphism classes}\\ \textit{of path connected}\\ \textit{coverings of } X\end{array}\right) \longrightarrow \left(\begin{array}{c}\textit{conjugacy classes}\\ \textit{of subgroups}\\ \textit{of } \pi_1(X, x_0)\end{array}\right)$$

given by $\Omega(p\colon T \to X) = p_*(\pi_1(T, \tilde{x}))$ for some $\tilde{x} \in p^{-1}(x_0)$ is a bijection.

21.2 Definition. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is an *equivalence of categories* if there exists a functor $G\colon \mathbf{D} \to \mathbf{C}$ for which the following conditions hold:

1) For each object $c \in \mathbf{C}$ there exists an isomorphism $\eta_c\colon c \to GF(c)$ such that for any morphism $f\colon c \to c'$ the following diagram commutes:

$$\begin{array}{ccc} c & \xrightarrow{\quad f \quad} & c' \\ {\scriptstyle \eta_c}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \eta_{c'}} \\ GF(c) & \xrightarrow[GF(f)]{} & GF(c') \end{array}$$

2) For each object $d \in \mathbf{D}$ there exists an isomorphism $\tau_d\colon d \to FG(d)$ such that for any morphism $g\colon d \to d'$ the following diagram commutes:

$$\begin{array}{ccc} d & \xrightarrow{\quad g \quad} & d' \\ {\scriptstyle \tau_d}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \tau_{d'}} \\ FG(d) & \xrightarrow[FG(g)]{} & FG(d') \end{array}$$

We will say that $\mathbf{C}$ and $\mathbf{D}$ are *equivalent categories* if there exists an equivalence $\mathbf{C} \to \mathbf{D}$.

21.3 Proposition. *A functor $F\colon \mathsf{C} \to \mathsf{D}$ is an equivalence of categories if and only if the following conditions hold.*

(i) For each object $d \in \mathsf{D}$ there exists an object $c \in \mathsf{C}$ such that $d \overset{\cong}{=} F(c)$.

(ii) For any objects $c, c' \in \mathsf{C}$ the map $\mathrm{Mor}_{\mathsf{C}}(c, c') \to \mathrm{Mor}_{\mathsf{D}}(F(c), F(c'))$ given by $f \mapsto F(f)$ is a bijection.

Proof. Exercise. □