2 | Categories and Functors

2.1 Definition. A *category* $\mathbf{C}$ consists of the following ingredients:

1) a class of *objects* $\mathrm{Ob}(\mathbf{C})$
2) for each pair of objects $c, c' \in \mathrm{Ob}(\mathbf{C})$ a set of *morphisms* $\mathrm{Mor}_{\mathbf{C}}(c, c')$
3) for each object $c \in \mathrm{Ob}(\mathbf{C})$ a distinguished *identity morphism* $\mathrm{id}_c \in \mathrm{Mor}_{\mathbf{C}}(c, c)$
4) for each triple of objects $c, c', c'' \in \mathrm{Ob}(\mathbf{C})$ a *composition of morphisms* function

$$\circ\colon \mathrm{Mor}_{\mathbf{C}}(c, c') \times \mathrm{Mor}_{\mathbf{C}}(c', c'') \to \mathrm{Mor}_{\mathbf{C}}(c, c'')$$

Moreover, the composition of morphisms satisfies the following conditions:

(i) $f \circ (g \circ h) = (f \circ g) \circ h$, whenever morphisms f, g, h are composable

(ii) if $f \in \mathrm{Mor}_{\mathbf{C}}(c, c')$ then $f \circ \mathrm{id}_c = f = \mathrm{id}_{c'} \circ f$.

2.8 Definition. Let $\mathbf{C}$, $\mathbf{D}$ be categories. A *(covariant) functor* $F\colon \mathbf{C} \to \mathbf{D}$ consists of

1) an assignment $F\colon \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$
2) for each $c, c' \in \mathrm{Ob}(\mathbf{C})$ a function

$$F\colon \mathrm{Mor}_{\mathbf{C}}(c, c') \to \mathrm{Mor}_{\mathbf{D}}(F(c), F(c'))$$

   such that $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$ for all $c \in \mathrm{Ob}(\mathbf{C})$ and $F(f \circ g) = F(f) \circ F(g)$ for each pair of composable morphisms f, g in $\mathbf{C}$.

2.11 Definition. Let C be a category. A morphism $f\colon c \to c'$ in C is an *isomorphism* if there exists a morphism $g\colon c' \to c$ such that $gf = \mathrm{id}_c$ and $fg = \mathrm{id}_{c'}$. In such case we say that g is the *inverse* of f and we write $g = f^{-1}$.

If there exists an isomorphism between $c, c' \in \mathsf{C}$ then we say that these objects are *isomorphic* and we write $c \cong c'$.

2.15 Proposition. *Let $F\colon \mathsf{C} \to \mathsf{D}$ be a functor. If $f\colon c \to c'$ is an isomorphism in C then $F(f)\colon F(c) \to F(c')$ is an isomorphism in D and $F(f)^{-1} = F(f^{-1})$.*

2.16 Corollary. *Let $F\colon \mathsf{C} \to \mathsf{D}$ be a functor and $c, c' \in \mathsf{C}$. If $F(c) \ncong F(c')$ then $c \ncong c'$.*