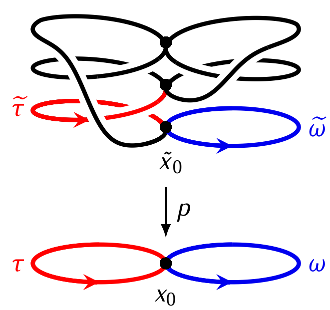
18 | Coverings and the Fundamental Group

18.1 Theorem. *Let $p\colon T \to X$ be a covering, let $x_0 \in X$ and let $\tilde{x}_0 \in p^{-1}(x_0)$.*

1) The homomorphism $p_*\colon \pi_1(T, \tilde{x}_0) \to \pi_1(X, x_0)$ is 1-1.

2) An element $[\omega] \in \pi_1(X, x_0)$ is in the subgroup $p_*(\pi_1(T, \tilde{x}_0)) \subseteq \pi_1(X, x_0)$ if and only if the lift $\widetilde{\omega}$ such that $\widetilde{\omega}(0) = \tilde{x}_0$ is a loop in T.



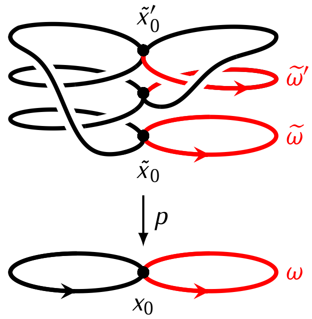
18.3 Proposition. *Let $p\colon T \to X$ be a covering, let $x_0 \in X$ and let $\tilde{x}_0 \in p^{-1}(x_0)$. Assume that ω_1 and ω_2 are paths in X such that $\omega_1(0) = \omega_2(0) = x_0$ and $\omega_1(1) = \omega_2(1)$. For $i = 1, 2$ let $\widetilde{\omega}_i\colon [0, 1] \to T$ be the lift of ω_i such that $\widetilde{\omega}_i(0) = \tilde{x}_0$. Then $\widetilde{\omega}_1(1) = \widetilde{\omega}_2(1)$ if and only if $[\omega_1 * \overline{\omega}_2] \in p_*(\pi_1(T, \tilde{x}_0))$.*

Proof. Exercise. □

18.4 Corollary. *Let $p\colon T \to X$ be a covering such that T is a path connected space, let $x_0 \in X$, and let $\tilde{x}_0 \in p^{-1}(x_0)$. Denote $H = p_*(\pi_1(T, \tilde{x}_0))$.*

1) For $i = 1, 2$ let ω_i be a loop in X based at x_0 and let $\widetilde{\omega}_i$ be the lift of ω such that $\widetilde{\omega}_i(0) = \tilde{x}_0$. We have $\widetilde{\omega}_1(1) = \widetilde{\omega}_2(1)$ if and only if $[\omega_1]H = [\omega_2]H$.

2) The index $[\pi_1(X, x_0) : H]$ is equal to the number of elements of the fiber $p^{-1}(x_0)$.

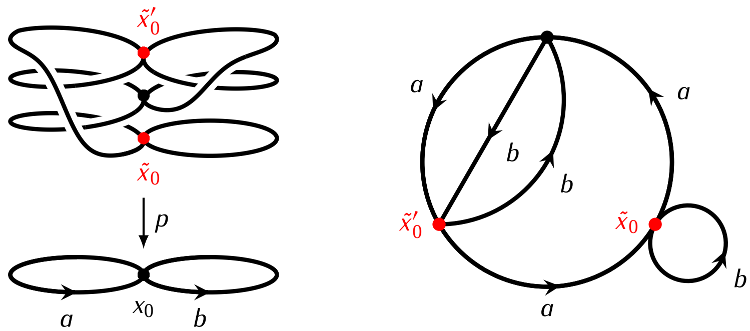


18.5 Proposition. *Let $p\colon T \to X$ be a covering such that T is a path connected space and let $x_0 \in X$.*

1) For any $\tilde{x}_0, \tilde{x}_0' \in p^{-1}(x_0)$ the subgroups $p_*(\pi_1(T, \tilde{x}_0))$ and $p_*(\pi_1(T, \tilde{x}_0'))$ of $\pi_1(X, x_0)$ are conjugate.

2) If $\tilde{x}_0 \in p^{-1}(x_0)$ and $H \subseteq \pi_1(X, x_0)$ is a subgroup conjugate to $p_*(\pi_1(T, \tilde{x}_0))$ then $H = p_*(\pi_1(T, \tilde{x}_0'))$ for some $\tilde{x}_0' \in p^{-1}(x_0)$.

18.6 Example.



18.7 Definition. Let $p\colon T \to X$ be a covering with a path connected total space T and let $x_0 \in X$. The covering p is *regular* if $p_*(\pi_1(T, \tilde{x}_0)) = p_*(\pi_1(T, \tilde{x}_0'))$ for any $\tilde{x}_0, \tilde{x}_0' \in p^{-1}(x_0)$.

18.8 Proposition. *Let $p\colon T \to X$ be a covering such that T is a path connected space and let $x_0 \in X$. The following conditions are equivalent:*

1) The covering p is regular.

2) For any $\tilde{x}_0 \in p^{-1}(x_0)$ the group $p_*(\pi_1(T, \tilde{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$.

3) Let ω be a loop in X based at x_0. If ω has a lift which is a loop then every lift of ω is a loop, and if ω has a lift which is an open path then every lift of ω is an open path.

Proof. Exercise. □

18.9 Proposition. *Every free group on two or more generators contains a free subgroup on an infinite number of generators.*