# 17 | Covering Spaces

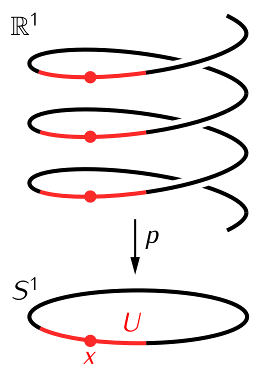
**17.1 Definition.** A map $p\colon T \to X$ is a *covering* of $X$ if for every point $x \in X$ there exists an open neighborhood $U_x \subseteq X$ and a homeomorphism $h_{U_x}\colon p^{-1}(U_x) \to U_x \times D_x$ where $D_x$ is some discrete space, such that the following diagram commutes:

$$\begin{tikzcd}
p^{-1}(U_x) \arrow[rr, "h_{U_x}"] \arrow[dr, "p"'] & & U_x \times D_x \arrow[dl, "\mathrm{pr}_1"] \\
& U_x &
\end{tikzcd}$$

Here $\mathrm{pr}_1\colon U_x \times D_x \to U_x$ is the projection map $\mathrm{pr}_1(y, d) = y$.

**17.3 Example.** Let $D$ be a discrete space. The projection map $\mathrm{pr}_1 \colon X \times D \to X$ is a covering of $X$. In this case the whole space $X$ is evenly covered. We say that $\mathrm{pr}_1 \colon X \times D \to X$ is a *trivial covering* of $X$.

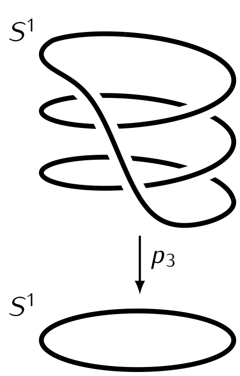
**17.4 Example.** Recall that the universal covering of $S^1$ is the map $p \colon \mathbb{R}^1 \to S^1$ given by $p(s) = (\cos 2\pi s, \sin 2\pi s)$. If $U \subseteq S^1$ is any open set such that $U \neq S^1$, then $p^{-1}(U)$ is evenly covered and $p^{-1}(U) \cong U \times \mathbb{Z}$ (exercise).




**17.5 Example.** Consider $S^1$ as a subset of the complex plane:

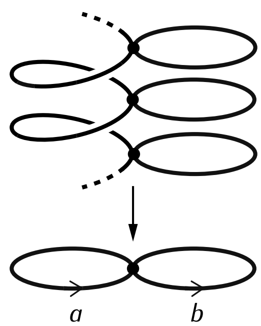
$$S^1 = \{z \in \mathbb{C} \mid \|z\| = 1\}$$

For $n = 1, 2, \ldots$ the map $p_n \colon S^1 \to S^1$ given by $p_n(z) = z^n$ is an $n$-fold covering of $S^1$. Similarly as in the case of the universal covering of $S^1$ any open set $U \subseteq S^1$ such that $U \neq S^1$ is evenly covered by $p_n$ (exercise).

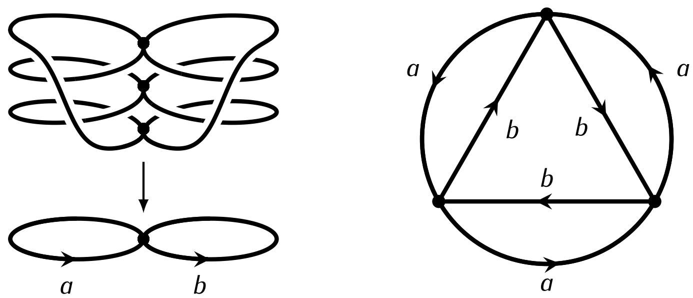


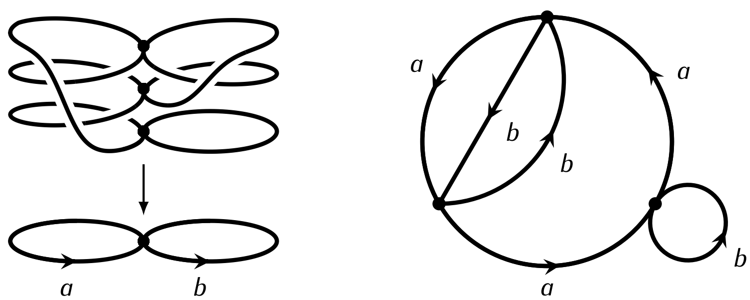
**17.6 Example.** If $p_1\colon T_1 \to X_1$ and $p_2\colon T_2 \to X_2$ are coverings then the map $p_1 \times p_2\colon T_1 \times T_2 \to X_1 \times X_2$ is also a covering (exercise). For example, starting with the universal covering $p\colon \mathbb{R}^1 \to S^1$ of the circle we obtain a covering $p \times p\colon \mathbb{R}^1 \times \mathbb{R}^1 \to S^1 \times S^1$ of the torus.

**17.7 Example.** Using the coverings of $S^1$ described above we can construct many coverings of $S^1 \vee S^1$. For example, here is a covering obtained by combining the universal covering over one copy of $S^1$ and a trivial covering over the second copy:



Here are two different 3-fold coverings of $S^1 \vee S^1$:



**17.8 Definition.** If $p\colon T \to X$ is a covering and $f\colon Y \to X$ is a map then a *lift* of $f$ is a map $\tilde{f}\colon Y \to T$ such that the following diagram commutes:

$$\begin{tikzcd} & T \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

**17.9 Theorem (Homotopy Lifting Property).** *Let $p\colon T \to X$ be a covering. Let $F\colon Y \times [0,1] \to X$ and $\tilde{f}\colon Y \times \{0\} \to T$ be functions satisfying $p\tilde{f} = F|_{Y\times\{0\}}$. There exists a function $\widetilde{F}\colon Y \times [0,1] \to T$ such that $p\widetilde{F} = F$ and $\widetilde{F}|_{Y\times\{0\}} = \tilde{f}$:*

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{f}"] \arrow[d, hook] & T \arrow[d, "p"] \\ Y \times [0,1] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & X \end{tikzcd} \tag{$*$}$$

*Moreover, such function $\widetilde{F}$ is unique.*

**17.10 Corollary.** *Let $p\colon T \to X$ be a covering. Let $x_0 \in X$, and let $\tilde{x}_0 \in T$ be a point such that $p(\tilde{x}_0) = x_0$.*

*1) For any path $\omega\colon [0,1] \to X$ such that $\omega(0) = x_0$ there exists a lift $\widetilde{\omega}\colon [0,1] \to T$ satisfying $\widetilde{\omega}(0) = \tilde{x}_0$. Moreover, such lift is unique.*

*2) Let $\omega, \tau\colon [0,1] \to X$ be paths such that $\omega(0) = \tau(0) = x_0$, $\omega(1) = \tau(1)$ and $\omega \simeq \tau$. If $\widetilde{\omega}$, $\widetilde{\tau}$ are lifts of $\omega$, $\tau$, respectively, such that $\widetilde{\omega}(0) = \widetilde{\tau}(0) = \tilde{x}_0$ then $\widetilde{\omega}(1) = \widetilde{\tau}(1)$ and $\widetilde{\omega} \simeq \widetilde{\tau}$.*

**17.11 Lemma.** *Let $p\colon T \to X$ be a covering, and let $\tilde{f}_1, \tilde{f}_2\colon Y \to T$ be two lifts of a map $f\colon Y \to X$. If $Y$ is a connected space and there exists $y_0 \in Y$ such that $\tilde{f}_1(y_0) = \tilde{f}_2(y_0)$ then $\tilde{f}_1(y) = \tilde{f}_2(y)$ for all $y \in Y$.*

$$\begin{tikzcd} & T \arrow[d, "p"] \\ Y \arrow[ur, shift left, "\tilde{f}_1"] \arrow[ur, shift right, "\tilde{f}_2"'] \arrow[r, "f"'] & X \end{tikzcd}$$

**17.12 Lemma.** *Let $p\colon T \to X$ be a covering. Let $F\colon Y \times [a,b] \to X$ and $\tilde{f}\colon Y \times \{a\} \to T$ be functions satisfying $p\tilde{f} = F|_{Y\times\{a\}}$. Assume also that $F(Y \times [a,b]) \subseteq U$ where $U \subseteq X$ is an evenly covered open set. There exists a function $\widetilde{F}\colon Y \times [a,b] \to T$ such that $p\widetilde{F} = F$ and $\widetilde{F}|_{Y\times\{a\}} = \tilde{f}$:*

$$\begin{tikzcd}
Y \times \{a\} \arrow[r, "\tilde{f}"] \arrow[d, hook] & T \arrow[d, "p"] \\
Y \times [a,b] \arrow[r, "F"'] \arrow[ur, dashed, "\widetilde{F}"] & X
\end{tikzcd}$$

*Proof of Theorem 17.9.* □