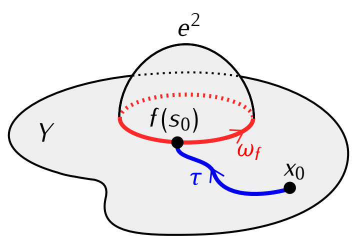
15 | Fundamental Group and 2-Cells

15.1 Theorem. *Let Y be a path connected space and let $X = Y \cup_f e^2$. Let $x_0 \in Y$ and let τ be a path in Y such that $\tau(0) = x_0$ and $\tau(1) = f(s_0)$. The homomorphism*

$$j_* \colon \pi_1(Y, x_0) \to \pi_1(X, x_0)$$

induced by the inclusion map $j\colon Y \to X$ is onto, and so $\pi_1(X, x_0) \cong \pi_1(Y, y_0)/\operatorname{Ker}(j_*)$. Moreover, $\operatorname{Ker}(j_*)$ is the normal subgroup of $\pi_1(Y, x_0)$ generated by the element $[\tau * \omega_f * \overline{\tau}]$.

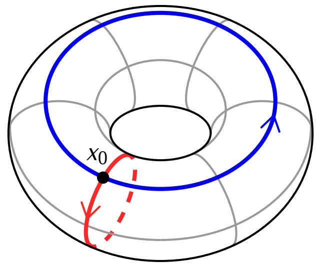


15.3 Theorem. *Let Y be a path connected space with basepoint $x_0 \in Y$ and let X be a space obtained by attaching to Y a collection of 2-cells: $X = Y \cup \{e_i^2\}_{i \in I}$. Let $f_i \colon S^1 \to X$ be the attaching map of the cell e_i^2 and let $\tau_i \colon [0,1] \to Y$ be a path such that $\tau_i(0) = x_0$ and $\tau_i(1) = f_i(s_0)$. The homomorphism*

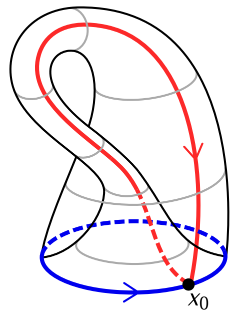
$$j_* \colon \pi_1(Y, x_0) \to \pi_1(X, x_0)$$

induced by the inclusion map $j \colon Y \to X$ is onto, and so $\pi_1(X, x_0) \cong \pi_1(Y, y_0)/\operatorname{Ker}(j_*)$. Moreover, $\operatorname{Ker}(j_*)$ is the normal subgroup of $\pi_1(Y, x_0)$ generated by set $\{[\tau_i * \omega_{f_i} * \overline{\tau}_i]\}_{i \in I}$.

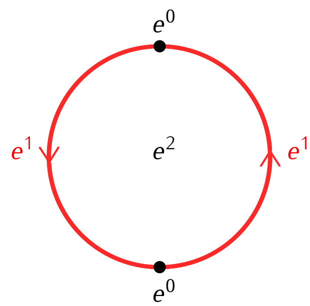
15.4 Example. Torus



15.5 Example. Klein bottle



15.6 Example. $\mathbb{RP}^2$



15.7 Note. Abelianization.

15.8 Theorem. *For any group G there exists a CW complex X such that $\pi_1(X) \cong G$.*