14 | Presentations of Groups

14.1 Definition. Free group generated by a set S

14.3 Note. We will say that a group G is free if G is isomorphic to the group $F(S)$ for some set S. Notice that by Theorem 13.13 the fundamental group of any 1-dimensional CW complex is free.

14.4 Theorem. *Let S be a set and G be a group. For any map of sets $f\colon S \to G$ there exists a unique homomorphism of groups $\bar{f}\colon F(S) \to G$ such that the following diagram commutes:*

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "i"'] & G \\
F(S) \arrow[ur, dashed, "\bar{f}"'] &
\end{tikzcd}$$

14.5 Definition. Let S be a set, and let R be a subset of elements of the free group $F(S)$. By $\langle S \mid R \rangle$ we denote the group given by

$$\langle S \mid R \rangle = F(S)/N$$

where N is the smallest normal subgroup of $F(S)$ such that $R \subseteq N$. We say that elements of S are *generators* of $\langle S \mid R \rangle$ and elements of R are *relations* in $\langle S \mid R \rangle$.

14.9 Definition. If G is a group and $G \cong \langle S \mid R \rangle$ for some set S and some $R \subseteq F(S)$ then we say that $\langle S \mid R \rangle$ is a *presentation* of G.

14.10 Definition. If a group G has a presentation $\langle S \mid R \rangle$ such that S is a finite set then we say that G is *finitely generated* and if it has a presentations such that both S and R are finite sets then we say that G is *finitely presented*.

14.11 Proposition. *Every group has a presentation.*

14.12 Note. 1) Every group has inifinitely many different presentations. For example

$$\mathbb{Z} \cong \langle a \rangle \cong \langle a, b \mid b \rangle \cong \langle a, b \mid ab^{-1} \rangle \cong \langle a, b \mid b^2, b^3 \rangle$$

2) In general if we know a presentation of a group it may be very difficult to say anything about the properties of the group (even if the group is trivial or not).