13 | Homotopy Extension Property

13.1 Definition. Let X be a topological space, and let $A \subseteq X$. The pair (X, A) has the *homotopy extension property* if any map

$$h\colon X \times \{0\} \cup A \times [0,1] \to Y$$

can be extended to a map $\bar{h}\colon X \times [0,1] \to Y$.

13.2 Proposition. *A pair (X, A) has the homotopy extension property if and only if $X \times \{0\} \cup A \times [0,1]$ is a retract of $X \times [0,1]$.*

Proof. Exercise. □

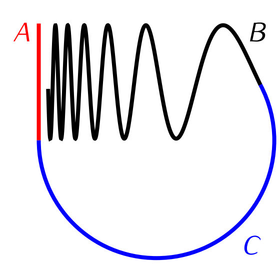
13.3 Proposition. *If a pair (X, A) has the homotopy extension property and X is a Hausdorff space then A is closed in X.*

Proof. Exercise. □

13.4 Proposition. *If a pair (X, A) has the homotopy extension property and the space A is contractible then the quotient map $q\colon X \to X/A$ is a homotopy equivalence.*

Proof. Exercise □

13.6 Example. Warsaw Curve:



13.7 Theorem. *Any relative CW complex* (X, Y) *has the homotopy extension property.*

13.8 Lemma. *For any* $n > 0$ *the pair* (D^n, S^{n-1}) *has the homotopy extension property.*

13.9 Proposition. *For any continuous function* $f\colon X \to Y$ *the pair* $(M_f, X \times \{0\})$ *has the homotopy extension property.*

Proof. Exercise. □

Proof of Lemma 13.8.

□

13.10 Lemma. *Let Y be any space an let $X = Y \cup \{e^n_\alpha\}_{\alpha \in I}$ be a space obtained from by attaching some number of n-cells to X. Then the pair (X, Y) has the homotopy extension property.*

13.11 Theorem. *If X is a path connected finite CW complex of dimension 1 then $X \simeq \bigvee_{i=1}^{n} S^1$ where*

$$n = \begin{pmatrix} \text{number of} \\ \text{1-cells of } X \end{pmatrix} - \begin{pmatrix} \text{number of} \\ \text{0-cells of } X \end{pmatrix} + 1$$

13.12 Corollary. *If X is a path connected finite CW complex of dimension 1 then $\pi_1(X) \cong *_{i=1}^{n}\mathbb{Z}$ where n is defined as in Theorem 13.11.*

Theorem 13.11 can be generalized to infinite 1-dimensional complexes:

13.13 Theorem. *If X is a path connected 1-dimensional CW complex then $X \simeq \bigvee_{I \in I} S^1$ for some set I. As a consequence $\pi_1(X) \cong *_{i \in I}\mathbb{Z}$.*

13.14 Note. Euler characteristic.