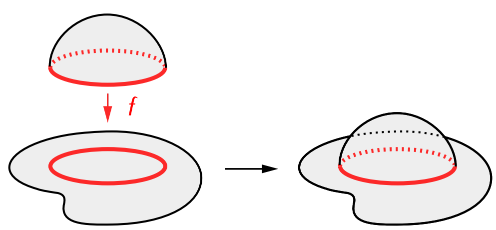
12 | CW Complexes

12.1 Definition. Let X be a space and let $f\colon S^{n-1} \to X$ be a continuous function. We say that a space Y is obtained by *attaching an n-cell* to X if $Y = X \sqcup D^n/\!\sim$ where $\sim$ is the equivalence relation given by $x \sim f(x)$ for all $x \in S^{n-1} \subseteq D^n$. We write $Y = X \cup_f e^n$.



12.2 Some terminology.

Here is some terminology associated to the operation of cell attachment:

- The map $f\colon S^{n-1} \to X$ is called the *attaching map* of the cell e^n.
- The map $\bar{f}\colon D^n \to X \sqcup D^n \to X \cup_f e^n$ is called the *characteristic map* of the cell e^n.
- The subspace $e^n = \bar{f}(D^n \smallsetminus S^{n-1}) \subseteq X \cup_f e^n$ is called the *open cell*.
- The subspace $\bar{e}^n = \bar{f}(D^n) \subseteq X \cup_f e^n$ is called the *closed cell*.

12.5 Lemma. *For any map $f\colon S^{n-1} \to X$ the space $X \cup_f e^n$ is homeomorphic to the mapping cone C_f.*

Proof. Exercise. $\square$

12.6 Proposition. *If $f, g\colon S^{n-1} \to X$ are maps such that $f \simeq g$ then $X \cup_f e^n \simeq X \cup_g e^n$.*

12.7 Definition. Let X be topological space and let $Y \subseteq X$. The pair (X, Y) is a *relative CW complex* if $X = \bigcup_{n=-1}^{\infty} X^{(n)}$ where

1) $X^{(-1)} = Y$;
2) for $n \geq 0$ the space $X^{(n)}$ is obtained by attaching n-cells to $X^{(n-1)}$;
3) the topology on X is defined so that a set $U \subseteq X$ is open if and only if $U \cap X^{(n)}$ is open in $X^{(n)}$ for all n.

12.8 Note. By part 3) of Definition 12.7 if (X, Y) is a relative CW complex then a function $f\colon X \to Z$ is continuous if and only if $f|_{X^{(n)}}\colon X^{(n)} \to Z$ is continuous for all $n \geq -1$.

12.9 Note. If (X, Y) is a relative CW complex then the space $X^{(n)}$ is called *the n-skeleton* of X.

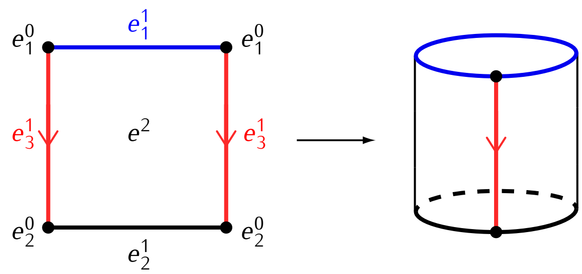
12.10 Definition. A *CW complex* is a space X such that $(X, \varnothing)$ is a relative CW complex.

12.11 Definition. 1) A CW complex X is *finite* if it consists of finitely many cells.

2) A CW complex X is *finite dimensional* if $X = X^{(n)}$ for some n.

3) The *dimension* of a CW complex X is defined by

$$\dim X = \begin{cases} \min\{n \mid X = X^{(n)}\} & \text{if } X \text{ is finite dimensional} \\ \infty & \text{otherwise} \end{cases}$$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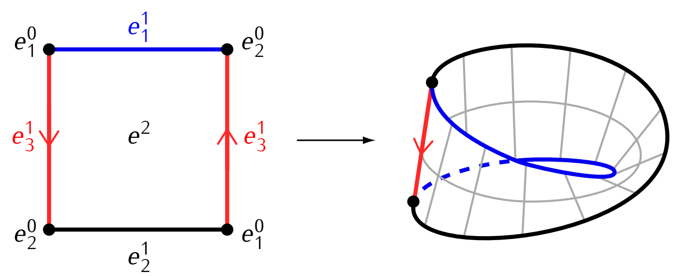
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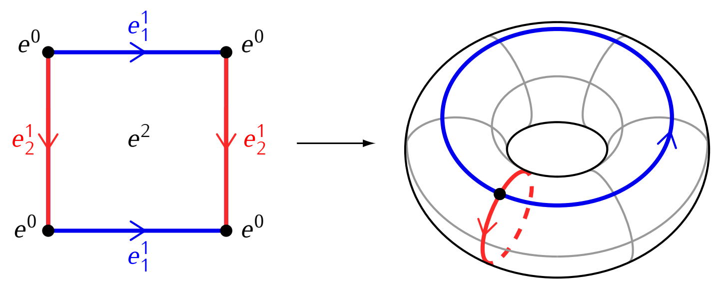
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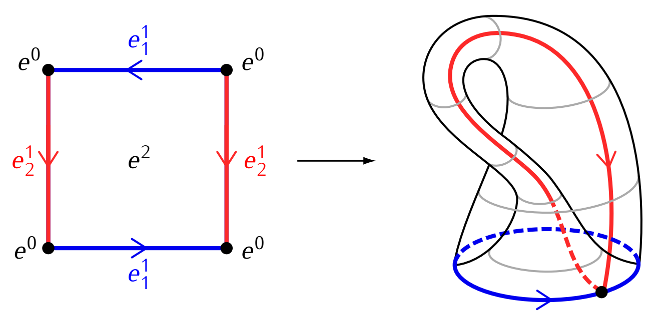
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12.17 Note. It is not true that every space can be given a structure of a CW complex.

12.18 Proposition. *1) Let (X, Y) be a relative CW complex. If $A \subseteq X$ is a compact set then A is closed in X and it has a non-empty intersection with finitely many open cells of X only.*

2) If X is a CW complex and $A \subseteq X$ is a closed set which has a non-empty intersection with only finitely many open cells of X then A is compact.

Proof. Exercise. □

12.19 Corollary. *A CW complex is compact if and only if it is a finite.*