10 | Pushouts and van Kampen's Theorem

10.1 Definition. Definition of a pushout.

10.3 Proposition. *Let* $c_1 \xleftarrow{f_1} c_0 \xrightarrow{f_2} c_2$ *be a diagram in a category* $\mathbf{C}$ *and let* $p, p' \in \mathbf{C}$*.*

1) If p *is a pushout of this diagram and* $p' \overset{\simeq}{=} p$ *then* p' *is also a pushout.*

2) Conversely, both p *and* p' *are pushouts of the above diagram then* $p \overset{\simeq}{=} p'$*.*

Pushouts of topological spaces.

10.5 Proposition. *For any diagram of topological spaces* $X_1 \xleftarrow{f_1} X_0 \xrightarrow{f_2} X_2$ *the pushout exists and it is given by*

$$\mathrm{colim}(X_1 \xleftarrow{f_1} X_0 \xrightarrow{f_2} X_2) = (X_1 \sqcup X_2)/\sim$$

where $\sim$ *is the equivalence relation defined by* $f_1(x) \sim f_2(x)$ *for all* $x \in X_0$*.*

Proof. Exercise □

10.9 Example. The following fact will be used later on. If X is a topological space and $U, V \subseteq X$ are open sets such that $X = U \cup V$ then we have a homeomorphism

$$X \cong \mathrm{colim}(U \longleftarrow U \cap V \longrightarrow V)$$

(exercise). Note that this is not true in general, if U, V are not open in X.

Pushouts of groups.

10.10 Definition. Free product of groups $G * H$.

10.13 Proposition. *For any diagram of groups* $G_1 \xleftarrow{f_1} G_0 \xrightarrow{f_2} G_2$ *the pushout exists and it is given by*

$$\operatorname{colim}(G_1 \xleftarrow{f_1} G_0 \xrightarrow{f_2} G_2) = (G_1 * G_2)/N$$

where N *is the normal subgroup of* $G_1 * G_2$ *generated by all elements of the form* $f_1(g)f_2(g)^{-1}$ *for* $g \in G_0$.

Proof. Exercise. □

10.17 van Kampen Theorem. *Let* (X, x_0) *be a pointed topological space and let* $U_1, U_2 \subseteq X$ *be open sets such that* $X = U_1 \cup U_2$*. If the sets* U_1*,* U_2*, and* $U_1 \cap U_2$ *are path connected and* $x_0 \in U_1 \cap U_2$ *then*

$$\pi_1(X, x_0) \cong \operatorname{colim}(\pi_1(U_1, x_0) \xleftarrow{i_{1*}} \pi_1(U_1 \cap U_2, x_0) \xrightarrow{i_{2*}} \pi_1(U_2, x_0))$$

where for $k = 1, 2$ *the homomorphism* i_{k*} *is induced by the inclusion map* $i_k \colon U_1 \cap U_2 \to U_k$*.*

10.19 Example. $\pi_1(S^1 \vee S^1)$

10.20 Lemma. *Let X be a space and let $U_1, U_2 \subseteq X$ be open sets such that $X = U_1 \cup U_2$ and $U_1, U_2, U_1 \cap U_2$ are path connected. If $\pi_1(U_1) \cong \{1\}$ and $\pi_1(U_2) \cong \{1\}$ then $\pi_1(X) \cong \{1\}$.*

10.21 Example. $\pi_1(S^n)$